General Physics: Electromagnetism, Correction 14

Exercise 1 :

At $t = 0$, a 12.0 V battery is connected in series with a 220 mH inductor and a total of 30 Ω resistance, as shown in the figure below.

1. What is the current at $t = 0$?
2. What is the time constant?
3. What is the maximum current?
4. How long will it take the current to reach half its maximum possible value?
5. At this instant, at what rate is energy being delivered by the battery and
6. At what rate is energy being stored in the inductor's magnetic field?

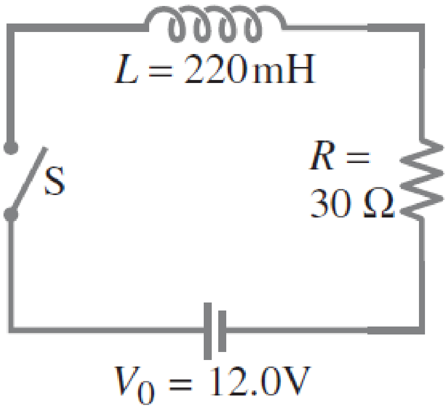

Figure 1: RL circuit.

Solution 1 :

We write the Kirchoff's law for the circuit, knowing that the voltage due to the self-inductance of the solenoid is given by $\varepsilon = -L dI/dt$:

$$V_0 - L\frac{dI}{dt} = IR. \tag{1}$$

The ordinary differential equation can be solved with separation of variables technique:

$$\int_{I(t=0)}^{I} \frac{dI'}{-V_0/R + I'} = -\int_0^t \frac{R}{L} dt'. \tag{2}$$

Where $I(t=0)=0$ is the current at $t=0$. The current is zero due to the presence of the inductor, which opposes to the change of the current (this is an instance of the Lenz law). The result is:

$$\log\left(\frac{V_0}{R}-I\right)-\log\frac{V_0}{R}=\frac{R}{L}t, \tag{3}$$

and then:

$$I(t)=\frac{V_0}{R}(1-e^{-\frac{R}{L}t}). \tag{4}$$

Therefore:

1. The current at $t=0$ is $I=0$ A
2. The time constant is $\tau=L/R=7.3\,\mathrm{ms}$
3. The maximum current is $I_{max}=V_0/R=0.4\,\mathrm{A}$
4. We compute the time $t_{1/2}$ when $I(t)=I_{max}/2$:

   $$\frac{V_0}{2R}=\frac{V_0}{R}(1-e^{-\frac{R}{L}t_{1/2}}) \tag{5}$$

   And we have $t_{1/2}=L\log 2/R=5\,\mathrm{ms}$.
5. At this instant, the energy rate delivered by the battery is given the power:

   $$P=I_{1/2}V=2.4\,\mathrm{W} \tag{6}$$

6. While the rate of energy stored in the inductor can be computed with the formula $U=\frac{1}{2}LI^2$. The rate will simply be the derivative of U:

   $$\frac{dU}{dt}=IL\frac{dI}{dt}=I(V_0-RI)=1.2\,\mathrm{W} \tag{7}$$

   Where we used the expression for the current derived before.

Exercise 2 :

In the circuit below, the switch S is closed at $t = 0$.

1. Write down the differential equations for the currents in the circuit and find the equivalent RL circuit;

   Hint: Inductors in parallel combine the same way as resistors in parallel.

2. Compute the expression of the currents $i_1(t)$ and $i_2(t)$ flowing in the two inductances at a generic time t.

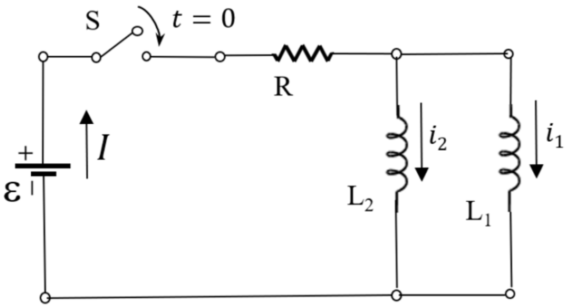

Figure 2: RL circuit.

Solution 2 :

1. Before the switch is closed, nothing happens since there is no current flowing in the circuit. From Kirchhoff rules, we can see that:

$$\begin{aligned} I &= i_1 + i_2 \\ \mathcal{E} - L_2 \cdot \frac{di_2}{dt} &= RI \\ L_1 \cdot \frac{d\left(i_1\right)}{dt} - L_2 \cdot \frac{d(i_2)}{dt} &= 0 \end{aligned} \tag{8}$$

   We can write $i_2 = I - i_1$ to get:

$$\begin{aligned} \mathcal{E} - L_2 \cdot \frac{d(I - i_1)}{dt} &= RI \\ L_1 \cdot \frac{d\left(i_1\right)}{dt} - L_2 \cdot \frac{d(I - i_1)}{dt} &= 0 \end{aligned} \tag{9}$$

   Then, we can get $\frac{di_1}{dt}$ from the second equation and plug it into the first one to get:

$$\mathcal{E} - L_2 \cdot \frac{dI}{dt} + \frac{L_2^2}{L_1 + L_2} \frac{dI}{dt} = RI. \tag{10}$$

Starting from this last equation, we can easily get:

$$\mathcal{E} - RI = \frac{L_1 \cdot L_2}{L_1 + L_2} \cdot \frac{dI}{dt}, \tag{11}$$

which turns out being the known differential equation of a LC circuit with an equivalent inductance:

$$L_{eq} = \frac{L_1 \cdot L_2}{L_1 + L_2}. \tag{12}$$

2. The time evolution of I is the typical one of a RL circuit:

$$I(t) = \frac{\mathcal{E}}{R}(1 - e^{-t/\tau}), \tag{13}$$

where the time constant is defined as:

$$\tau = \frac{L_{eq}}{R} = \frac{L_1 \cdot L_2}{R \cdot (L_1 + L_2)}. \tag{14}$$

For the current i_1, from the second one of Eq.(9), we get:

$$\frac{di_1(t)}{dt} = \frac{L_2}{L_1 + L_2} \cdot \frac{dI}{dt} = \frac{\mathcal{E}}{L_1} e^{-t/\tau}, \tag{15}$$

whose solution is:

$$i_1(t) = \int \frac{di_1(t')}{dt'} dt' = -\frac{L_2}{R \cdot (L_1 + L_2)} \cdot \mathcal{E} \cdot e^{-t/\tau} + \text{const} \tag{16}$$

At $t = 0$, we have that $i_1(0) = 0$, condition from which we can find the integration constant:

$$\text{const} = \frac{L_2}{R(L_1 + L_2)} \cdot \mathcal{E}. \tag{17}$$

And, in conclusion, with this value of constant, we can finally get the expressions for the time-evolution of the two currents $i_1(t)$ and $i_2(t)$:

$$\begin{aligned} i_1(t) &= \frac{L_2}{R(L_1 + L_2)} \cdot \mathcal{E} \cdot (1 - e^{-t/\tau}) \\ i_2(t) &= \frac{L_1}{R(L_1 + L_2)} \cdot \mathcal{E} \cdot (1 - e^{-t/\tau}). \end{aligned} \tag{18}$$

Exercise 3 :

A 1200 pF capacitor is fully charged by a 500 V dc power supply. It is disconnected from the power supply and is connected, at $t = 0$, to a 75 mH inductor. Determine:

1. The initial charge on the capacitor;
2. The maximum current;
3. The frequency f and period T of oscillation;
4. The total energy oscillating in the system.

Solution 3 :

1. The 500V power supply, before being disconnected, charged the capacitor to a charge of
$$Q_0 = CV = \left(1.2 \times 10^{-9}\text{ F}\right)(500\text{ V}) = 6.0 \times 10^{-7}\text{C}. \tag{19}$$
2. The maximum current, $I_{\max}$, is
$$I_{\max} = \omega Q_0 = \frac{Q_0}{\sqrt{LC}} = \frac{(6.0 \times 10^{-7}\text{C})}{\sqrt{(0.075\text{H})\left(1.2 \times 10^{-9}\text{ F}\right)}} = 63\text{ mA}. \tag{20}$$
3. The frequency reads
$$f = \frac{\omega}{2\pi} = \frac{1}{(2\pi\sqrt{LC})} = 17\text{ kHz}, \tag{21}$$
and the period T is
$$T = \frac{1}{f} = 6.0 \times 10^{-5}\text{ s}. \tag{22}$$
4. Finally the total energy is given by
$$U = \frac{Q_0^2}{2C} = \frac{\left(6.0 \times 10^{-7}\text{C}\right)^2}{2\left(1.2 \times 10^{-9}\text{ F}\right)} = 1.5 \times 10^{-4}\text{ J}. \tag{23}$$

Exercise 4 :

The switch in the circuit shown in the figure below is held in position [a] for 2 seconds. The switch is then instantly moved to position [b] (without interrupting the electric current through the coil). At time $t = 2$ s, the capacitor carries no charge.

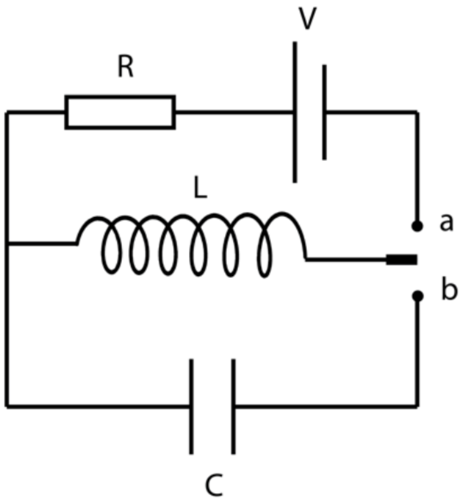

Figure 3: RL circuit.

In position [b] of the switch:

1. Calculate the electric current through the coil at time $t = 2$s.
2. Calculate the frequency of oscillation.
3. Determine the charge $Q(t)$ on the capacitor C as a function of time t.
4. Determine the current $I(t)$ as a function of time t.

Numerical application : $V = 0.2$ V, $R = 0.03\ \Omega$, $L = 54$ mH, $C = 3.2\ \mu$F.

Solution 4 :

1. When the switch is kept in position [a], the current circulates only in the loop formed by the battery, resistance R, and the inductor L. Applying Kirchhoff's second law to this loop, we get the following differential equation:

$$RI + L\frac{dI}{dt} = V.$$

To solve it, write

$$L\frac{dI}{dt} = V - RI.$$

If we integrate this equation we get

$$\int \frac{dI}{V - RI} = \int \frac{dt}{L} \quad \Longrightarrow \quad -\frac{1}{R}\log(V - RI) = \frac{t}{L} + const,$$

which leads to [since $I(0) = 0$]

$$I = \frac{V}{R}(1 - e^{-Rt/L}).$$

For $t = 2$ s, we obtain $I = 4.5\,\mathrm{A}$, which is the amplitude of the current.

2. After the switch is moved to position [b], oscillations appear in the circuit formed by the inductor L and the capacitor C. The angular frequency of oscillation is $\omega_0 = \frac{1}{\sqrt{LC}}$. The frequency is therefore:

$$f = \frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{LC}} = 382.9\,\mathrm{Hz}.$$

3. Applying Kirchhoff's second law to the loop formed by the inductor and the capacitor, we get the following differential equation:

$$\frac{Q(t)}{C} + L\frac{dI(t)}{dt} = 0,$$

which leads to the following differential equation for $Q(t)$:

$$\frac{Q(t)}{C} + L\frac{d^2Q(t)}{dt^2} = 0.$$

The solution is:

$$Q(t) = Q_0 \sin\left(\frac{t}{\sqrt{LC}}\right),$$

where we use the sine function instead of the cosine because $Q(0) = 0$. This allows us to determine the frequency of oscillation $\omega_0 = 2406\,\mathrm{rad\,s^{-1}}$ and $f_0 = 383\,\mathrm{Hz}$.

4. The current $I(t)$ is related to $dQ(t)/dt$ as follows:

$$I(t) = \frac{dQ(t)}{dt} = Q_0 \frac{1}{\sqrt{LC}} \cos\left(\frac{t}{\sqrt{LC}}\right) = Q_0\omega_0 \cos\left(\omega_0 t\right).$$

Exercise 5 :

A conducting bar moves at a constant velocity v along two straight rails that form an angle α, in the presence of a uniform magnetic field B perpendicular to the triangle formed by the bar and the rails. The resistance of the bar is negligible, while the rails have a cross-sectional area S and are made of a material with resistivity ρ.

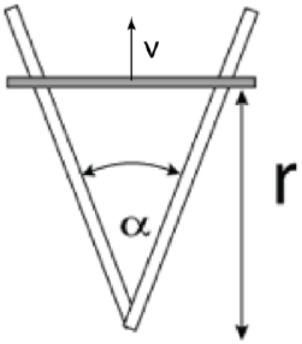

Figure 4: RL circuit.

1. Determine the amplitude of the current I induced in the bar when the distance between the bar and the vertex is r.
2. Determine the magnetic force on the moving bar as a function of time.

Initial conditions: $t = 0$ s and $r = 0$ cm. Numerical application $r = 12$ cm, $S = 1$ mm^2, $B = 0.2$ T, $\alpha = 50°$, $\rho = 1.5 \cdot 10^{-6}\ \Omega \cdot m$, $v = 30$ cm/s and $t = 0.4$ s.

Solution 5 :

We start by calculating some quantities that will be necessary later, namely the distance between the bar and the vertex, r, and the length of the part of the bar enclosed in the rails, l:

$$r(t) = vt, \quad l(r) = 2r \cdot \tan\left(\frac{\alpha}{2}\right) = 2vt \cdot \tan\left(\frac{\alpha}{2}\right).$$

From this we can compute the distance d from the vertex and the conjunction between the rail and the bar, by using Pitagora formula

$$d = \sqrt{(\frac{l}{2})^2 + r^2} = r \cdot \sqrt{\tan^2\left(\frac{\alpha}{2}\right) + 1} = \frac{r}{\cos\left(\frac{\alpha}{2}\right)}.$$

We can now compute the magnetic flux through the circuit when the magnetic field is perpendicular to the plane of the figure and the induced voltage, by applying the Farady's law:

$$\Phi_B(t) = B \cdot l(t) \cdot \frac{r(t)}{2} = Bv^2t^2 \cdot \tan\left(\frac{\alpha}{2}\right),$$
$$V_{\text{ind}}(t) = -\frac{\partial \Phi_B(t)}{\partial t} = -2Bv^2t \cdot \tan\left(\frac{\alpha}{2}\right).$$

Notice that the induced voltage remains time dependent. We need also to compute the resistance of the two rails, by using the usual definition of R

$$R(t) = \frac{2\rho}{S \cdot \cos\left(\frac{\alpha}{2}\right)} vt.$$

1. At this point it becomes straightforward to compute the magnitude of the induced current as a function of position r, which is simply given by

$$I(t) = \frac{|V_{\text{ind}}(t)|}{R(t)} = \frac{BvS}{\rho} \cdot \tan\left(\frac{\alpha}{2}\right) \cdot \cos\left(\frac{\alpha}{2}\right) = \frac{BvS}{\rho} \cdot \sin\left(\frac{\alpha}{2}\right) = 17\,\text{mA}.$$

Notice that while V_{ind} and R are both time dependent, the induced current is instead time independent.

2. The magnetic force induced on the moving bar is instead given by

$$F(t) = -\frac{2B^2v^2tS}{\rho} \cdot \sin\left(\frac{\alpha}{2}\right) \cdot \tan\left(\frac{\alpha}{2}\right) = 3.78 \times 10^{-4}\,\text{N}$$

Exercise 6 :

A 30 pF air-gap capacitor has two circular plates of area $A = 100\,\mathrm{cm}^2$. It is charged by a 70 V battery through a $2.0\,\Omega$ resistor. In the instant the battery is connected, the electric field between the plates is changing most rapidly. In this instant, calculate:

1. The current into the plates;
2. The rate of change of electric field between the plates;
3. Determine the magnetic field induced between the plates. Assume $\vec{E}$ is uniform between the plates at any instant and is zero at all points beyond the edges of the plates.

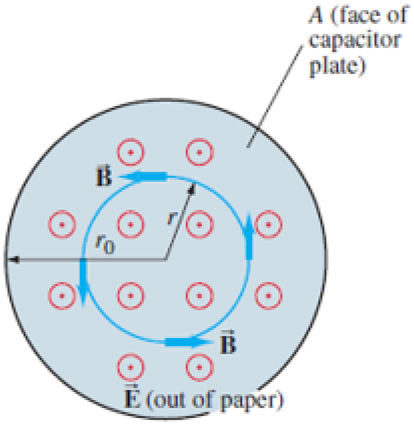

Figure 5: Frontal view of the circular plate of a parallel plate capacitor. $\vec{E}$ between plates points out toward viewer, lines of $\vec{B}$ are circles.

Solution 6 :

In this exercise, we are interested in the behaviour of a charging capacitor. To describe its behaviour, we apply Kirchoff's law to write down the differential equation

$$V = RI(t) + \frac{Q(t)}{C} = R\frac{dQ(t)}{dt} + \frac{Q(t)}{C}. \tag{24}$$

By integrating it we have

$$\int \frac{dQ}{V - Q/C} = \int \frac{dt}{R} \quad \Longrightarrow \quad -C\log(V - Q/C) = \frac{t}{R} + const, \tag{25}$$

which leads to [since $Q(0) = 0$]

$$Q(t) = VC(1 - e^{-t/RC}). \tag{26}$$

1. We can find the current in the capacitor plates at time $t = 0$, from the definition of the current, $I = \frac{dQ}{dt}$. We can then express the current as:

$$I = \left.\frac{dQ}{dt}\right|_{t=0} = \left.\frac{\varepsilon C}{RC}e^{-t/RC}\right|_{t=0} = \frac{\varepsilon}{R} = \frac{70\,\mathrm{V}}{2\,\Omega} = 35\,\mathrm{A} \tag{27}$$

2. To find the rate of change of the electric field, let's start from the electric field for a capacitor, defined as:
$$E = \frac{\sigma}{\epsilon_0} \tag{28}$$
where $\sigma = Q/A$ is the surface charge for a surface A. Now by taking the derivative if this E-field one can find:
$$\frac{dE}{dt} = \frac{d}{dt}\frac{\sigma}{\epsilon_0} = \frac{1}{A\epsilon_0}\frac{dQ}{dt}\bigg|_{t=0} = \frac{I}{A\epsilon_0} = 4\,\mathrm{V\,m^{-1}\,s} \tag{29}$$
where we have used the results of question 1.

3. We are now encountering a time varying electric field. From the generalized Ampère law (Ampère-Maxwell law), we have that,
$$\oint_C \vec{B}\cdot d\vec{\ell} = \mu_0 I_{\text{enc}} + \mu_0\epsilon_0\frac{d\Phi_E}{dt} \tag{30}$$
where $\Phi_E = \int_S \vec{E}\cdot d\vec{\sigma}$ is the electric flux. We see that a time varying Electric field must produce a magnetic field. From this geometry, we can imagine that the resulting magnetic field will have a circular shape perpendicular to the electric field. We can then take a circular Ampère loop parallel to and between the capacitor plates. We will have two cases as function of the radius of the loop, a case where it is smaller than the radius of the capacitor plates and a case where it will be bigger. For both cases there will be no enclosed currents, $I_{\text{end}} = 0$, hence:
$$\oint_C \vec{B}d\vec{l} = \mu_0\epsilon_0\frac{d\Phi_E}{dt} \tag{31}$$
We first start with the configuration where the radius of the loop, r, is smaller than the radius of the capacitor plate, r_0 ($r < r_0$):
$$\oint_C \vec{B}d\vec{l} = \mu_0\epsilon_0\frac{d\Phi_E}{dt} \tag{32}$$
$$B(2\pi r) = \mu_0\epsilon_0\frac{d}{dt}\int_S \vec{E}.d\vec{S} \tag{33}$$
$$B(2\pi r) = \mu_0\epsilon_0\frac{d}{dt}E(\pi r^2) \tag{34}$$
$$B = \frac{\mu_0\epsilon_0 r}{2}\frac{dE}{dt} \tag{35}$$
where $\Phi_E = \int_S \vec{E}.d\vec{S} = E(\pi r^2)$.

We have then to analyze the case for $r > r_0$.
$$\oint_C \vec{B}d\vec{l} = \mu_0\epsilon_0\frac{d\Phi_E}{dt} \tag{36}$$
$$B(2\pi r) = \mu_0\epsilon_0\frac{d}{dt}\int_S \vec{E}.d\vec{S} \tag{37}$$
$$B(2\pi r) = \mu_0\epsilon_0\frac{d}{dt}E(\pi r_0^2) \tag{38}$$
$$B = \frac{\mu_0\epsilon_0 r_0^2}{2r}\frac{dE}{dt} \tag{39}$$

In this case, the flux $\Phi_E = E(\pi r_0^2)$, since $\vec{E} = 0$ outside the plates.

B has its maximum value at $r = r_0$ which, from either relation above (using $r_0 = \sqrt{A/\pi} = 5.6\,\mathrm{cm}$), is:

$$B = \frac{\mu_0 \epsilon_0 r_0}{2} \frac{dE}{dt} = 1.2 \times 10^{-4}\,\mathrm{T}. \tag{40}$$

This is a very small field and lasts only briefly (the time constant $RC = 6 \times 10^{-11}\,\mathrm{s}$) and so would be very difficult to measure.

Extra:
Let us write the magnetic field outside the capacitor plates in terms of the current that leaves the plates. The electric field between the plates is $E = \sigma/\epsilon_0 = Q/(\epsilon_0 A)$, as we saw in question 2, $\frac{dE}{dt} = I/(\epsilon_0 A)$. Hence B for $r > r_0$ is,

$$B = \frac{\mu_0 \epsilon_0 r_0^2}{2r} \frac{dE}{dt} = \frac{\mu_0 \epsilon_0 r_0^2}{2r} \frac{I}{\epsilon_0 \pi r_0^2} = \frac{\mu_0 I}{2\pi r}. \tag{41}$$

This is the same formula for the field that surrounds a wire. Thus the B field outside the capacitor is the same as that outside the wire. In other words, the magnetic field produced by the changing electric field between the plates is the same as that produced by the current in the wire.

Exercise 7 :

An induction stove contains a coil of copper wire underneath the ceramic plate, the "burner" (a burner that never gets hot). When a cooking pot is placed on top of it, an alternating perpendicular magnetic field is applied. The resulting oscillating magnetic field induce a magnetic flux changing in time, producing Eddy currents in the pot. These currents will heat up the pot due to Joule heating.

The heating power of an induction stove, designed for Switzerland, is 2 kW. What is the power of this stove if it is used in the USA with the same metal pan?

Hint:

-
    - CH: $V_{\text{eff,CH}} = 240$ V, $\omega_{\text{CH}} = 50$ Hz;
    - USA: $V_{\text{eff,USA}} = 115$ V, $\omega_{\text{USA}} = 60$ Hz;
- It can be useful to define an effective voltage as follow, $V_{\text{eff}} = \frac{1}{T}\int_t^{t+T}(V_0 \sin(\omega t'))^2 dt' = \frac{V_0}{\sqrt{2}}$.

Solution 7 :

The heating power of the pan comes from the induced current $I_{\text{AC,ind}}$. This current is induced by the time-varying magnetic field generated by a coil in the stove. The magnetic field is generated in the coil because a voltage $V_{\text{AC,CH}}$ is applied to the coil and this voltage creates a current according to:

$$I_{\text{AC,CH}} = \frac{V_{\text{AC,CH}}}{R_c}, \tag{42}$$

where R_c is the coil resistance and the label "CH" stands for "Switzerland". So, the picture is the following (in Switzerland):

$$V_{\text{AC,CH}}(t) = V_{0,\text{CH}} \sin(\omega_{\text{CH}}(t)) \tag{43}$$

is applied to the coil;

$$I_{\text{AC,CH}}(t) = \frac{V_{0,\text{CH}}}{R_c} \sin(\omega_{\text{CH}}(t)) \tag{44}$$

flows through the coil;

$$B_{\text{CH}}(t) = \mu_0 n \frac{V_{0,\text{CH}}}{R_c} \sin(\omega_{CH}(t)) \tag{45}$$

is generated by the coil. The flux of B_{CH} through the surface of the pan changes and a current $I_{\text{ind,CH}}$ is induced in the pan according to:

$$I_{\text{ind,CH}} = \frac{emf}{R_{\text{pan}}} = -\frac{1}{R_{\text{pan}}}\frac{\partial}{\partial t}\oint_{S_{\text{pan}}} \vec{B}\cdot d\vec{S} = -\frac{1}{R_{\text{pan}}} A_{\text{pan}} \frac{\partial \vec{B}}{\partial t} = -\frac{A_{\text{pan}}}{R_{\text{pan}}}\mu_0 n \frac{V_{0,\text{CH}}}{R_c}\omega_{\text{CH}} \cos(\omega_{\text{CH}} t). \tag{46}$$

$A_{\text{pan}}, R_{\text{pan}}, \mu_0$ and n are constant which do not depend on the state the stove is used, so we can write:

$$I_{\text{ind,CH}} = C V_{0,\text{CH}} \omega_{\text{CH}} \cos(\omega_{\text{CH}} t). \tag{47}$$

We know that in Switzerland the voltage provided in the domestic networks is 240 V, but this is an effective value, i.e. the root-mean-square value:

$$V_{\text{eff}} = \sqrt{\frac{1}{T}\int_t^{t+T}(V_0 \sin(\omega t'))^2 dt'} = \frac{V_0}{\sqrt{2}}. \tag{48}$$

This effective value is related to the peak value of the AC signal, but it is independent on the country, so we can still write:

$$I_{\mathrm{ind,eff,CH}} = C V_{\mathrm{eff,CH}} \omega_{\mathrm{CH}}. \tag{49}$$

The effective power is:

$$P_{\mathrm{eff,CH}} = R_{\mathrm{pan}} I^2_{\mathrm{ind,eff,CH}} = C' V^2_{\mathrm{eff,CH}} \omega^2_{\mathrm{CH}}, \tag{50}$$

where C' is just a new constant. In the US, both V_{eff} and ω are different:

- CH: $V_{\mathrm{eff,CH}} = 240$ V, $\omega_{\mathrm{CH}} = 50$ Hz;
- USA: $V_{\mathrm{eff,USA}} = 115$ V, $\omega_{\mathrm{USA}} = 60$ Hz;

So, if we compare:

$$P_{\mathrm{eff,USA}} = R_{\mathrm{pan}} I^2_{\mathrm{ind,eff,USA}} = C' V^2_{\mathrm{eff,USA}} \omega^2_{\mathrm{USA}}, \tag{51}$$

we get:

$$\frac{P_{\mathrm{eff,USA}}}{P_{\mathrm{eff,CH}}} = \frac{V^2_{\mathrm{eff,USA}} \omega^2_{USA}}{V^2_{\mathrm{eff,CH}} \omega^2_{\mathrm{CH}}} \tag{52}$$

From which we conclude that:

$$P_{\mathrm{eff,USA}} = P_{eff,CH} \frac{V^2_{\mathrm{eff,USA}} \omega^2_{\mathrm{USA}}}{V^2_{\mathrm{eff,CH}} \omega^2_{\mathrm{CH}}} = 0.33 P_{\mathrm{eff,CH}} = 0.66\ kW. \tag{53}$$